Solution to Assignment 2

Supplementary Problems

Note the notations. These problems are valid in all dimensions. Hence we do not use (x, y) to denote a generic point as we do in $\mathbb{R}^2$. Instead, here $\mathbf{x}$ or $\mathbf{p}$ are used to denote a generic point in $\mathbb{R}^n$.

1. Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(\mathbf{x}) = 1$ for $\mathbf{x} \in S$ and $\chi_S(\mathbf{x}) = 0$ otherwise. Prove the following identities:

   (a) $\chi_{A\cup B} \le \chi_A + \chi_B$.

   (b) $\chi_{A\cup B} = \chi_A + \chi_B$ if and only if $A \bigcap B = \phi$, that is, A and B are disjoint.

   (c) $\chi_{A\cap B} = \chi_A \chi_B$.

   Solution. (a) For $\mathbf{x} \in A \cup B, x$ must belong either to A or B. Hence $\chi_{A\cup B}(\mathbf{x}) = 1 \le \chi_A(\mathbf{x}) + \chi_B(\mathbf{x})$. On the other hand, when $\mathbf{x}$ does not belong to $A \cup B$, $\chi_{A\cup B}(\mathbf{x}) = 0$ and the inequality clearly holds.

   (b) and (c) are left to you.

2. Let f be integrable in a domain D which satisfies $A \le f \le B$ for two numbers A and B everywhere. Show that
$$A|D| \le \int_D f \le B|D| \ ,$$
   where $|D|$ is the "area" of D.

   Solution. By assumption, $B - f(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in D$. Hence
$$\begin{aligned} 0 &\le \int_D (B - f) \\ &= \int_D B - \int_D f \quad \text{(linearity)} \\ &= B|D| - \int_D f \ , \end{aligned}$$
   and the second inequality follows. The first one can be proved by using $f(\mathbf{x}) - A \ge 0$. (The area is better understood as the n-dimensional volume.)